ISSN 2229-5518

Fifth postulate of Euclid and the non-Euclidean geometries. Implications with the spacetime.

Jairo Eduardo Márquez Díaz

Abstract- This article shows the results of the study conducted on Euclidean geometry, in particular the fifth postulate, which led to the emergence of non-Euclidean geometries. As a research methodology, we proceeded to carry out a study on the mathematical and geometric modeling that characterizes this type of geometry, establishing its differences framed in the theories formulated by its discoverers, simulating some of them, in order to show the spatial representation of the so-called geodesic shapes and curves. In the same way, the impact and diverse applications of these geometries in other sciences are exposed in a general way, in particular cosmology, where space is conjugated with time, generating another type of space-time metrics such as the Riemannian one, which allows explanation and support to theories such as the general relativity of Einstein, and other physico-theoretical models related to quantum physics, giving way to new approaches on the characteristics and incidence of matter and energy in the macro and micro context of the universe.

Index Terms - Cosmology, Elliptical Geometry, Spacetime, Euclidean Geometry, Fifth Postulate of Euclides, Geodesic, Hyperbolic Geometry.

1. INTRODUCTION

Euclidean geometry marked the path of mathematics and science for more than two thousand years, where great thinkers and scientists shaped scientific culture under its foundations, accompanied by philosophy as support for their approaches, inventions and discoveries. Without detracting from the great contribution made by Euclid, his postulates are no longer used in a generalized manner, particularly in sciences such as astrophysics, cosmology, in mathematical logic and in some approaches of quantum physics, among other scientific fields. This is due to the development of algebra, whose use of numbers to measure things acquired a fundamental importance. [1] In addition, with the development of algebra, unlike the postulates of Euclid that are characterized by being axiomatic, another type of mathematics was developed based on slogans, corollaries, theorems and even conjectures, much more structured and demonstrable, where the mathematical logic has played a fundamental role, not only in the field of mathematics, but in other disciplines such as science and engineering, for example, electronics, telecommunications and computer science, vital for the technological development of the contemporary society.

Euclidean geometry as such belongs to a physical world, and this has been conceived and taken for millennia, giving man the possibility of creating from the simplest constructions, to majestic architectural monuments and sculptures that have lasted to this day, just like it has served as a construct for the advancement of mathematics and science today. In this sense, nature and the mathematics itself showed its other side, in which about two centuries ago, the first proposals were proposed that refute the fifth postulate of Euclid, giving way to new types of geometry, which eventually converged to the creation of non-Euclidean or meta-mathematical geometry and mathematical logic. With this new type of geometry, natural phenomena are discovered and explained at cosmic and subatomic scales, which still amaze man, because of their complexity, dynamics and mathematical foundation, where the time variable is integrated with space, marking in this way a breaking point with Euclidean geometry.

In 13 volumes with the name of Elements, Euclid met the geometric knowledge of his time (late fourth century BC and early III a.c.). [2] Particularizing in his great work, the first four postulates of Euclid enunciate aspects of geometry whose demonstration is immediate and intuitive and do not present any discussion, so they are cited as a reference, like this:

Postulate I: For all points P and Q different, there is a single line l that passes through P and Q. It can be defined more informally as, for any point you can lead a line to any other point.

Fig. 1. Postulate I of Euclid. $\forall P \neq Q$, $\exists l$, a single line that passes through P and Q.

Postulate II: For all segments AB and CD, there is a point E, such that B is between points A and E, and segment CD is congruent (≅) with segments BE, as shown in figure 2. This the postulate can be defined more informally, as a limited right can be prolonged indefinitely by right.

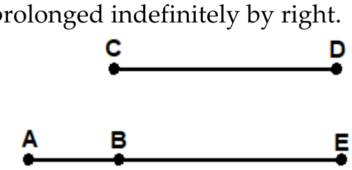


Fig. 2. Second postulate of Euclid. $\forall \overrightarrow{AB}$ *and* $\overrightarrow{CD}, \exists E$, such that B is between A and E, and $\overrightarrow{CD} \cong \overrightarrow{BE}$.

- *Jairo E. Márquez D. Systems Engineer, Mathematician and Physicist. Specialist in University Teaching, Specialist in Bioethics, Specialist in Actuarial, Specialist in Cyberdefense. Master in Bioethics, Master in Business Information Security, Candidate for Doctor in Education. Teacher researcher and leader of the research groups S@r@ and Nanosistemas Udec Chía, Cundinamarca University, Colombia. Email. jemarquez@ucundinamarca.edu.co. ORCID: https://orcid.org/0000-0001-6118-3865*

ISSN 2229-5518

Postulate III: For all points O and P different, there is a circumference between O and the radius OP. It can be defined more informally as a circle can be described from any center and any distance.

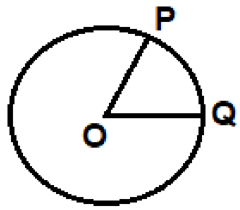


Fig. 3. The third postulate of Euclid is derived directly from the set theory for a set of points P, such that $\overrightarrow{OP} \cong \overrightarrow{OQ}$.

Postulate IV: All right angles are congruent with each other.

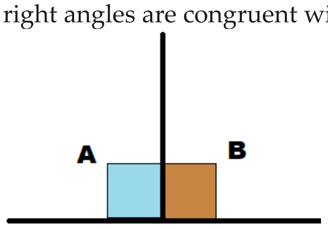


Fig. 4. All the right angles are equal, then ∢A = ∢B.

2. FIFTH POSTULATE OF EUCLIDES

There is a historical point of discussion that revolves around the fifth postulate of Euclid (or postulate of the parallels). Euclid uses parallels for the first time, in book I, prop. 27, by showing that if one line intersects two others in such a way that the two alternate angles are equal, the lines are parallel. [3] Another similar definition states that if a secant cuts two straight lines forming side angles whose sum is less than two straight, the two long enough lines are cut on this same side.

This postulate in particular showed to be not as intuitive as Euclides believed, so from the beginning he presented discussions about its validity. The problem of this postulate in particular, lies in the fact that many mathematicians and geometers throughout history, have considered that its demonstration is feasible, and therefore, should not be considered as a postulate. In this sense, when observing the evolution of geometry in history, which is relatively recent, we can thank Euclides for raising a gap in mathematical knowledge through this postulate, since it gave way to new approaches to geometry such as metamathematical, which involves a whole new development of mathematics and modern geometry, giving impetus to other scientific disciplines, as will be shown throughout this article.

There are other statements equivalent to the previous ones, some more understandable than others, which are summarized by Lucas [4] as follows:

1. For every line l and for every point P that is not above l, there is a single line m through P that is parallel to l, as shown in figure 5.

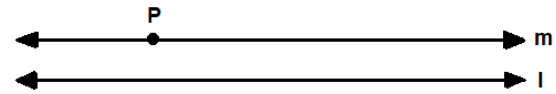


Fig. 5. The lines l and m are parallel to each other. The demonstration of this postulate can not be done empirically, because as you can see, you can only draw segments of finite lines, but not all of them.

2. There is a pair of straight lines in which all points of one are at the same distance from each other.
3. There is a pair of similar non-congruent triangles.
4. If in a quadrilateral a pair of opposite sides are equal and the angles adjacent to the third side are straight, then the other two angles are also straight.
5. If in a quadrilateral three angles are straight, then the fourth is also straight.
6. There is at least one triangle in which the sum of its three angles is equal to two straight.
7. For a point located within an angle of less than 60 degrees, a straight line that cuts on both sides of the angle can always be drawn.
8. A circumference can be passed through any three non-collinear points.
9. There is no upper limit to the area of a triangle.

In addition to the above, Vittone [5] quotes Legendre (1794), which states that the fifth postulate is also equivalent to the Pythagorean theorem, since it is a fact that the sum of the internal angles of a triangle is 180 degrees and the fact that there are triangles similar to a given one.

Due to the diverse interpretations for which this postulate has been lent, it gave much to talk about for a long time, beginning because it was affirmed at the time that it was a theorem that could be deduced from the first four, which was not and is not true. The interesting thing about this postulate is that when we put it into practice, we come across natural physical elements that this affirmation leaves between. For example, by logical inspection, it is inferred that if two lines are extended and cut, then they are not parallel; however, if the line segments are not cut, you can not be sure that there is no cut point at infinity. Under this assumption, the only demonstrable resource is through indirect reasoning, in which criteria different from the given definition are used. Thus, for a long time geometers and mathematicians tried without a resounding success, because many demonstrations were based on tacit assumptions related to the postulate itself, violating the logical rule of cyclic reasoning.

Of the great geometers, mathematicians and philosophers who stand out in the crusade to rebut the fifth postulate of Euclid without any success, but who paid the ground for others who did, are: Proclo, Wallis, Vitale,

ISSN 2229-5518

Playfair, Lambert and Lagrange. As there is an exception to every rule, this corresponds to Gerolamo Saccheri (1667-1733), who was very close to refuting this postulate, even to raise the emergence of non-Euclidean geometry, as stated by A. Trigo [6], in his article the fifth postulate of Euclid ... and the geometry of the universe:

"... Saccheri found results that seemed to contradict common sense and ended his work, without realizing that the anti-intuitive does not necessarily have to be antilogical or unnatural. Therefore, Saccheri had at hand the creation of non-Euclidean geometries, but he did not succeed because he was absolutely convinced that only Euclidean geometry could exist. "

The complete mathematical and geometrical demonstration about the approach made by Saccheri can be found in the text of J. Gray [3], which shows how close he was to refute the fifth postulate of Euclid, but also, to raise the geometries not Euclidean in demonstrative terms. This contribution of Saccheri did not go unnoticed, because it set a precedent to the academic community of that time, on the possibility that this postulate presented certain irregularities when space was not flat.

3. NON-EUCLIDIAN GEOMETRIES

3.1 Hyperbolic geometry

The Hungarian mathematician János Bolyai (1802-1860) restated the fifth postulate giving it a totally different approach to the Euclidean, defining it as follows: From a point outside a line, infinite lines can be drawn parallel to the given one. This contribution was underestimated by the mathematical community of that time and for which it hoped to give its approval and support, the mathematician Gauss, so it was relegated to oblivion for several years.

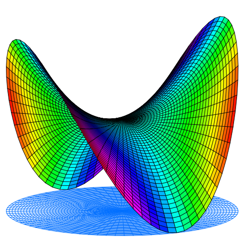
Parallel to the development of Bolyai, Nikolai Ivanovich Lobachevsky (1793-1856), formulated and published his work under the name "*Theory of parallels*", as follows: Given a line and a point outside it, at least they can be draw two parallels to the line through the point, which shows that the sum of the angles of a triangle is less than 180º (which for the Euclidean case equals exactly 180º), and that the intersection of two parallel planes contains parallel lines to each other.

It is important to note that the scale factor is fundamental for this type of affirmation, since if an observer is on the hyperbolic plane, he will not notice any difference between it and a Euclidean one. Figure 6 illustrates a hyperbolic plane, represented by the saddle type geometry. This geometry in particular, is one of the approaches of what has come to be called as the local geometry of the universe, characterized in that the curvature of the latter is negative, an aspect that will be discussed later.

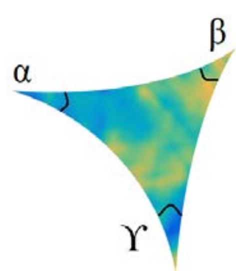
Johann Heinrich Lambert (1728-1777) independent of Bolyai and Lobachevsky, deduced by means of mathematical formulations about triangles projected in a hyperbolic geometry, that the sum of the interior angles are always less than 180º or π radians. For this condition to occur, one of these triangles must meet the following condition:

$$\big(\pi - (\alpha + \beta + \gamma)\big) = kA_{\alpha\beta\gamma} \quad (1)$$

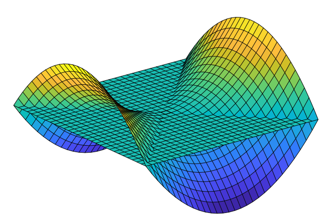
Where α, β and ϒ represent the internal angles of the triangle; $A_{\alpha\beta\Upsilon}$ represents the total area of the triangle, and k, is a positive proportional constant, which is related to the curvature of the hyperbolic space in which the triangle is located, as seen in figure 6. In general terms, build a model for this type of geometry involves changing the way you want to measure on a plane, which differs completely from the Euclidean metric.



(a)



(b)



(c)

Fig. 6. The first figure (a) represents the hyperbolic plane of the Bolyai-Lobachevsky geometry, from which a triangle with hyperbolic geometry is drawn on the chair (b), whose sum of the angles α, β and ϒ is lower at 180º. Note that below the hyperbolic plane is the Euclidean plane, where the sum of the angles is exactly 180º. (c) When making the superposition between the Euclidean and hyperbolic plane, the difference is notorious, since when projecting a triangle in the plane and the curved surface, the total sum of the angles in both planes will not be equal to 180º.

The work of Lambert had no importance in his time, only until the year 1820 when Bolyai and Lobachevsky, published independently their theories, which established the existence of another type of geometry. This test was expected to be given permanent the emergence of non-Euclidean or hyperbolic geometry, but had to wait thirty years until the work of Carl Friedrich Gauss after his death was published in 1855, in which part of them he spoke precisely of non-Euclidean geometry. It is important to mention that after this discovery there are controversial

ISSN 2229-5518

aspects about the bearing and development of Bolyai, Lobachevsky [7] and Gauss regarding their respective theories, which although not mentioned in the present article, the story in many cases has been unfair in giving unique credit to Gauss. Montesinos [8] makes a quite complete analysis on this subject, in which he cites that Gauss began at a young age to deal with the problem of parallels, which, like Saccheri and Lambert, obtained deductions in the hypothesis of the negation of the axiom of parallels independently. Similarly, in this work a compendium is made about Bolyai and Lobachevsky in terms of their contributions to non-Euclidean geometry and their academic and friendship relationship with Gauss, in which a quite interesting historical exhibition is made.

Taking up the theme, with Gauss publications, attention to the issue of non-Euclidean geometry, where the works of Bolyai and Lobachevsky were mentioned in 1866-1867 by the mathematician Richard Baltzer (1818-1887), was paid in shortly after it was becoming aware by the mathematical community about the transcendence of this new geometry in the physical world. [9] With the hyperbolic geometry, it was given way to related geometric developments, such as that of Klein and the pseudosphere, which are built on curved spaces; which play an extremely important role in explaining the general theory of Einstein's relativity.

One of the first mathematicians to study in depth the subject of hyperbolic geometry on a surface, was Beltrami (1869), raising the pseudosphere [10], obtained from a surface of revolution when a tractrix environment turns his asymptote, as seen in figure 7 (a). This rotation generates a negative Gauss curvature [11], which is constant, where each of the points on the surface is a saddle point. Márquez [12] mentions that Beltrami showed how the metric in the pseudosphere can be transferred to the unit disk and that the singularity of the pseudosphere corresponds to a horoscope in the hyperbolic plane.

Another example of hyperbolic geometry is applied in projective geometry, through the Klein-Beltrami model; which consists of an open disc in which the straight lines are Euclidean strings whose ends are at the frontier of the disc. [12]

(a) (b)

Fig. 7. Graphical representation of two types of hyperbolic geometry, the pseudosphere (a), and (b) the Poincaré disk, which in this case generates *n* projections, presenting fractal characteristics as a whole.

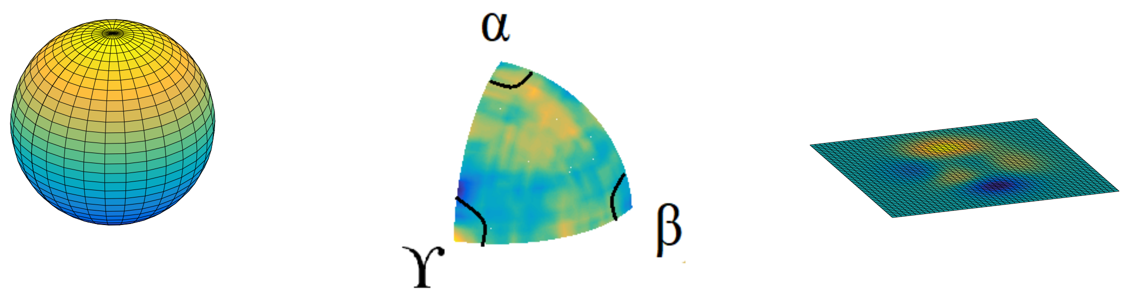
There are other models related to hyperbolic geometry such as the Poincaré disk or conformal disk (which analyzes the angle of parallelism and the horocycles, as shown in Figure 7 (b)), the Lorentz model or hyperboloid (which uses a hyperboloid sheet of revolution) and the Poincaré upper half-plane model, in which concepts and properties that belong to Euclidean geometry are related, such as: vertical lines, isometries (a representative case in this respect are the Möbius transformations), double ratio, translations, rotations, reflections, distances, lengths, and area of triangles, among others. It is recommended to review on these topics, in particular the Poincaré disk because it has greater applications in hyperbolic geometry in the Kisbye document [13].

3.2 Riemann's geometry

Another type of geometry that contravened the fifth postulate of Euclid, was raised by Bernhard Riemann (1826-1866, student of Gauss) defining it as elliptical geometry, which unlike the geometry of Bolyai-Lobachevsky, is characterized because the lines are not infinite but closed, therefore, the sum of the interior angles is greater than 180º. This geometry of positive constant curvature, is characterized because there are no parallel lines from an exterior point, therefore, it is not possible to draw any. For Riemann, constructing this type of geometry involved using a variety of elements defined by coordinates described by Euclidean infinitesimal metrics, in which arc segments are taken that define each coordinate point in space; which are represented by a positive quadratic form *ds* like:

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j \qquad (2)$$

In general terms, the most representative model of this type of *n-dimensional* geometry is the *n-sphere*, as illustrated in figure 8.



(a) (b) (c)

Fig. 8. Riemman elliptical geometry. In (a) a sphere representing an elliptical Riemman geometry is represented. The characteristic characteristic of this type of geometry is that the sum of the internal angles α, β and Υ is greater than 180º, where the triangle (b) belongs to the elliptical surface. Regarding the Euclidean plane (c), the difference is notorious, so the geodesics in both types of large-scale geometry differ completely.

Riemman distinguished the unlimited of the infinite in a geometry of this type, in which he defines that just as a curve can have no end (to be of unlimited extension) being finite in magnitude - a circle, for example - so the space can be unlimited but not necessarily infinite. [2] Under this statement, it is inferred that one can have a constant positive curvature, and therefore, have a finite radius.

The elliptical geometry as well as the hyperbolic one, are characterized by the scale factor to differentiate from the Euclidean geometry, since if an observer is in an elliptical plane like the surface of the Earth, he will not notice any difference between an east flat and one Euclidean. Therefore, in this context we speak of the geodesic lines, as illustrated in figure 8 - sphere (a) and euclidean plane (b) -, these types of planes have notable differences and one of them is precisely the fifth postulate.

A fundamental application of Riemann's elliptical geometry is in Einstein's theory of relativity, in which the four-dimensional metric of the universe is applied. In the words of B. Lewis: *"In Einstein's general theory of relativity, the geometry of space is a Riemannian geometry. Light travels through geodesics and the curvature of space is a function of the nature of the matter that composes it. "*[14] This type of geometry plays a fundamental role in modern and relativistic physics, where the object of study is spacetime. For this purpose, the spaces of constant curvature are proposed by means of the Riemann curvature tensor, which is mathematically expressed as:

$$R_{ijkl} = C\left(g_{il}g_{jk} - g_{ik}g_{jl}\right) \qquad (3)$$

Where g_{ij} represents the metric tensor of rank 2 in curvilinear coordinates [15], to which is proportionally associated to the tensor of Ricci R_{ij} and the scalar curvature S respectively:

$$R_{ij} = (n-1)Cg_{ij} \qquad (4)$$

$$S = n(n-1)C \qquad (5)$$

Where n represents the dimension of space.

The Riemann curvature tensor can be defined according to the Levi-Civita connection; which is applied in the general theory of relativity as a differentiable semi-Riemannian manifold M, and a metric tensor g of signature *(3,1)*, which in general terms is expressed as the pair *(M, g)*, such as it is exposed below.

Perdigão do Carmo [16] states that the function $g_{ij} = g_{ji}$, is called the local representation of the Riemannian metric (or the g_{ij} of the metric) in the coordinate system x: $U \subset \mathbb{R}^n \to M$. What indicates that a differentiable manifold with a given Riemannian metric will be called a Riemannian manifold? This way of conceiving space has evolved with the treatment given at the beginning of the 20th century in the works of the Italian mathematicians M. Ricci (1853-1925) and T. Levi-Civita (1873-1941) up to the notion that today is called as a Riemannian variety. [17]

Riemann suggests that the existence of Euclidean geometries can be considered when the dimension of these are infinitesimally small, therefore, the value of the curvature of space can differ. In this sense, the mathematical formalism of the curvature tensor is introduced, which allows demonstrating the other exposed geometries, including the Riemannian own. Under these approaches, Einstein uses the curvature tensor to explain his general theory of relativity in terms of the curvature behavior of spacetime geometry, which today is known as a gravitational field, and which is typical of bodies that have a large amount of mass and/or energy. Another application of the Riemann geometry is about the Levi-Civita connection, which allows to study the geodesics, curvatures and other elements of this geometry. The definition of this connection is relatively simple, as explained by Lafuente [18]:

Let *(M, g)* be a Semi-Riemannian variety. A linear connection ∇ in *M* is said to be compatible with the metric *g*, if $\nabla g = 0$, or equivalently, for all *X, Y, Z* $\in \Xi(M)$ we have:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad (6)$$

The condition in local coordinates is equivalent to $\varphi = (u^1, \dots, u^m)$ using the symbols of Christoffel is written as:

$$g_{ij;k} = -\Gamma^h_{ki} g_{hj} - \Gamma^h_{kj} g_{ih} = 0 \quad (7)$$

The functions $\Gamma^h_{ki} g_{hj} = \Gamma^h_{kj} g_{ih}$ are the Christoffel symbols of the first kind and completely determine the original Christoffel symbols Γ^k_{ij} in which the inverse matrix (g^{hk}) of the (g_{ij}) is used:

$$\Gamma^k_{ij} = \Gamma_{hij} g^{hk} \quad (8)$$

It is then that:

$$\frac{\partial g_{ij}}{\partial u^k} = \frac{\partial}{\partial u^k} g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \Gamma_{kij} + \Gamma_{kji} \quad (9)$$

Now, considering that $\Gamma_{kij} = \Gamma_{ikj}$ it is concluded that,

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u_j} + \frac{\partial g_{jk}}{\partial u_i} + \frac{\partial g_{ij}}{\partial u_k}\right) \text{ Where } i, j, k = 1,2 \quad (10)$$

With this result it is proved that there is a connection ∇ compatible with *g*.

As it is appreciated, the mathematical construction that is used in the Riemann geometry is quite wide and complex in some senses, that derives in the explanation of other physical-mathematical models, own of the differential geometry and the quantum physics, such as: connections and covariant derivative, varieties and Riemannian connections, lengths of curves and volumes, and pseudometric, among others.

3.3 Geodetic

A subject that is not alien to the geometries mentioned are the geodesics, whose importance lies in that it is applied in curved spaces. The geodesics in essence are described by lines through which a body or particle moves in spacetime, as shown in Figure 9.

Mathematically it is proposed as a curve ϒ that is defined in a range that belongs to the real numbers, whose speed is parallel, in which the following condition is fulfilled:

$$\frac{D_\gamma}{dt}\gamma' = 0 \quad (11)$$

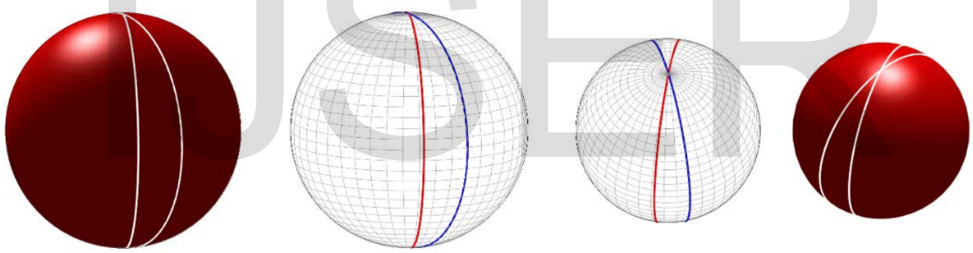


Fig. 9. Representation of the geodesic lines projected on a spherical surface, in which two parallel lines are highlighted. As it is observed, in graphical terms the fifth postulate of Euclides lacks validity, because when extending in a finitely big space the two lines are cut in two points.

What the equation (11) implies, is that in a geometric space there is a regular curve, where it is said that there is a geodesic if ϒ' is parallel. To this definition, the following theorem is added:

For all $x \in M$, $u \in T_x M$, there exists a single maximal geodesic *ϒ* such that *ϒ(0) = x*, *ϒ'(0) = u*. [19]

This type of geodesic describes related straight lines traveled with constant speed, that is to say, that does not present detectable accelerations from the surface. The condition of being geodesic can be written as a system of ordinary differential equations of the second order. [20] Now, since the curvature tensor is canceled, any straight line is a geodesic, where its second derivative is equal to zero $(\ddot{\tau} = \ddot{x} = \ddot{y} = \ddot{z})$. In relativistic terms, the representation of these geodetic lines for a particle that moves in a four-dimensional space (x_o, y_o, z_o, t) is the following:

$$\tau(t) = \tau_o + \frac{t}{\beta} \qquad x(t) = x_o + \frac{tv_x}{\beta} \qquad y(t) = y_o + \frac{tv_y}{\beta}$$
$$z(t) = z_o + \frac{tv_z}{\beta} \quad (12)$$

Where $\beta = \sqrt{1 - \left(\frac{v}{c}\right)^2}$ representing a relativistic coefficient; (v_x, v_y, v_z) are the spatial components of the velocity of a particle and *t* is the proper time of the particle within its reference system.

In more colloquial terms, the geodesic represents a line of length *L* that joins two points on a curved surface or Riemannian variety. This definition represents a geodetic Euclidean $(\mathbb{R}^n, g_o)$.

There is another type of geodesics related to the Riemannian varieties, such as: those of spherical type *(Sn (1), g)*, projective space $\mathbb{R}P^n$ and in the hyperbolic plane with the model of the semiplane. [21] It should be noted that a variety is curved if the parallel transport of a vector through a closed curve results in a different vector when returning at the exit point. [22]

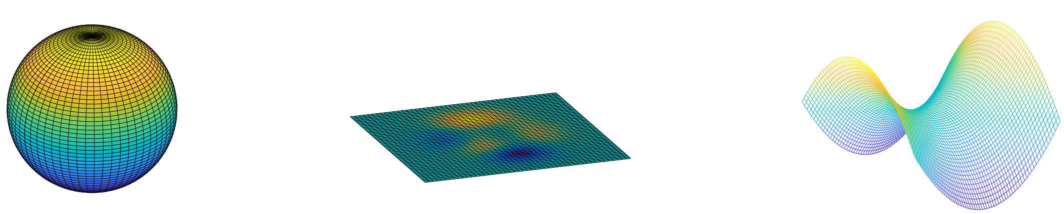
Based on the above, it follows that the geodesics have a direct relationship with the topology of spacetime and its causal structure in bodies of great mass and/or energy in the universe. Thus, the theory of causality is dedicated to the study of the relationships between the different points of a Lorentz variety, that is, it studies the points that can be joined by causal or temporal curves. [23] In this sense, the theory arises about the existence of closed curves, which would imply that certain particles could move between the past, present and future, thus enabling the time travel of matter and Energy. This is a physical phenomenon, which for some years has gone from being considered as science fiction, to being treated as a subject of critical research, as much for research centers as for governments, since behind it research is moving like the theory of multiple universes, teleportation and quantum computing.

Another proposal revolves around the existence of the Cauchy hypersurfaces, which, unlike the previous theory, would allow knowing the state of a particle in the future. For the case of critical mass bodies such as neutron stars and black holes, there are incomplete geodesics present in the so-called space-time singularities, where the deformation of space is so great, that time tends to zero as matter and energy approaches the center of the singularity, breaking the geometric structure of spacetime. For these critical cases in particular, the Euclidean geometry does not apply, meanwhile the non-Euclidean geometries present an anomalous and irregular behavior, due to the atypical conditions of these physical systems, since it is speculated that the number of dimensions is reduced with respect to the surrounding dimensions of our universe and that could increase when it reaches the open or closed gravitational singularity, which derive in other types of singularities, such as: temporal, spatial or naked, each with its own metric, which depends on the quantity of matter and energy engaged in these stellar physical systems.

For the case where the dimensions are increased in a physical system, the Riemann tensor is used and for a three-dimensional system, the Ricci curvature tensor. Therefore, dimensionality plays an extremely important role in the geometry of relativistic spacetime and quantum mechanics, because although there has been talk about macro systems, at the quantum level, geometry plays a fundamental role in explaining the behavior of matter. subatomic scales, since the number of dimensions goes up to 11 (which can decrease to 10 depending on the intensity of the coupling interactions, whether they are large or small) [24], this is the case for the *theory M* that originates from the string theory (needs 7 spatial dimensions), which generalizes the quantum field theory, and which is one of the most accepted at present, since it makes it possible to explain the cosmology of the *D-branes* in terms of the assumption that our universe would be part of other universes, acting as a bubble with its own physical laws. That is, reference is being made to the theory of parallel or multiple universes (multiverses), where some of them would present totally different behaviors to our universe, either in terms of the type of matter and energy that compose it, such as the physical laws that govern them.

4 GEOMETRY AND COSMOLOGY

Non-Euclidean geometry explains phenomena of the universe that it is not possible to do with Euclidean geometry, starting with the curvature of the universe, in which one does not know for sure if it is Euclidean (flat), elliptical (positive), or hyperbolic (negative), as illustrated in Figure 10. Due to the uncertainty regarding the geometry of the universe, its end is still uncertain, although it is still in a process of expansion or inflation, as has been verified in an observational manner in which many scientists affirm that the universe under this scheme is open. Also, there are detractors, who claim that the universe will reach a point where it will stop expanding and start a process of contraction or big crunch, so it is speculated that it would be closed, even speculating that it would present a cyclic explosion process - implosion eternally.



$\Omega > 1$ $\Omega = 1$ $\Omega < 1$

Fig. 9. The local geometry of the universe is roughly determined by using the omega curvature constant (Ω). Figure (a) represents a spherical or Riemannian universe, characterized by having a positive curvature, which indicates that it is a closed universe with a high density of matter. Figure (b) represents a flat or Euclidean universe of curvature 0, whose density of matter is critical. Figure (c) represents a hyperbolic universe, with a negative curvature, so it is considered an open universe with a low density of matter.

The geometry of the universe depends on the amount of mass and energy it has, which apparently has much more than what is assumed, which is called as dark matter, which exceeds in several orders the visible mass of the universe. This type of dark matter does not emit or absorb electromagnetic radiation, so it is not possible to even measure it and/or detect it directly with current technology. As an additional fact, the energy density of the universe is divided into three main components: 68.3% dark energy, 26.8% dark matter and only 4.9% baryonic matter (or visible). [25]

Dark matter is attributed to the existence of mass cohesion in galaxies and stars, which allows them to stay together and form groups or clusters. Although there are other candidates that would add mass, such as: antimatter galaxies, exoplanets with dark matter [26], supermassive black holes and exotic high energy dark matter particles such as WIMP (Weakly Interacting Massive Particles) which hypothetically would be part of the dark matter, the truth is that we do not know yet what it is composed of. It should be noted that it is speculated that WIMPs would resemble neutrinos, but with a higher mass. [27] From what is known with certainty until now, is that there is some kind of matter around the galaxies and stars that keeps them cohesive and that eventually act as a brake to the expansion of the universe.

Although at present it has not been demonstrated that there is any direct interaction of dark matter with ordinary matter, physical models predict that it is possible, and it is through gravity, because, after all, it was because of medium of it that dark matter was detected. Therefore, there would be disintegrations that would be induced by the curvature of space, the manifestation of gravity according to Albert Einstein's theory of general relativity. [28] Fernández [29] states that Einstein was the one who deduced that spacetime acquires curvature in the presence of material masses with given energy distributions, according to a set of equations of the form:

$$Curvature = G * Energy\ density \quad (11)$$

This equation is equivalent to:

$$G_{ij} = \frac{8\pi G}{c^4} T_{ij} \qquad (12)$$

Where G_{ij} is the Einstein tensor, which represents the curvature or geometry of spacetime; G represents the universal gravitation constant, which measures the rigidity of space-time, that is, its resistance to be curved by the presence of energy or its equivalent, matter; c the speed of light and T_{ij}, is the energy-moment tensor, which represents the distribution of energies and their flows present in space-time. This approach has been confirmed repeatedly, through direct and indirect observations on certain stellar phenomena.

Another physical aspect that belongs to non-Euclidean geometry is the curvature of the universe, which is directly related to spacetime, as demonstrated by Hernann Minkowski (1864-1909) with his metric:

$$ds^2 = dx^2 + dy^2 + dz^2 - cdt^2 \quad (13)$$

This metric represents a Lorentzian variety [30] of four dimensions (three spatial dimensions and one temporal, and one null and isomorphic curvature, which is used to describe the physical phenomena related to the special theory of relativity, the quantum mechanics of holes blacks, the quantum theory of gravity in 4D (a space-time 3 + 1) [31], the Einstein-Rosen bridge, string theory, and even wormholes, for example, in the case of Black holes, a research published in Nature by Graham and colleagues [32], states that the quasar *PG 1302-102* could harbor a binary system of supermassive black holes separated by a few hundred light years away. is related to the variation of the brightness of the quasar, whose period observed by the CRTS (Catalina Real-Time Transient Survey) is 5 years.

The variations cited, are related to strong distortions of spacetime, therefore, the physical laws in this type of system must have a totally irregular behavior to the surrounding universe, and therefore, the geometry implicit in this type of system as well. The reason for citing this cosmic system lies precisely in its spatio-temporal geometry, which although local, extends for several light years, so it is speculated that in this particular area the distortions of space and time are critical, and that in theory should be observable. Under this assumption, it is stated that a space-time field would be formed, which would behave like a bubble of time, due to the strong gravitational fields present there. Although speculative this statement, what we want to show, is that the non-Euclidean geometries in this type of systems adopt a limit behavior, where mathematics and physics still fall short in their explanation.

Another type of metrics to be noted for its relevance in astrophysics and cosmology, are the Schwarzschild metric and the Friedman-Lemaitre-Robertson-Walker metric: the first is characterized by the fact that it is applied to a space-time geometry of a body spherical, which is considered that its angular momentum is zero and is also isolated, an example of this is a static black hole [33]. As discussed by Polanco and Arrtche [34], the Schwarzschild metric [35] for a star described in terms of the characteristic line element in spherical spatial coordinates, can be written in the form,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 sen^2\theta d\varphi^2 \qquad (14)$$

Where $f(r) = 1 - \frac{2m}{r}$. The parameter $m = \frac{GM}{c^2}$ represents the geometric mass, whose units are of length that depends on the universal gravitation constant *G*, the speed of light *c* and the mass of the central body or star; which is responsible for deforming the spacetime and subsequent formation of the event or event horizon, which is estimated to be formed at a distance of $r = rs = 2m$, where r_s is the Schwarzschild radius ($2GM/c^2$). This radius is understood, as the critical distance where matter once it enters, has no return, including light, entering what is called as singularity.

The event horizon is not unique, because it can be evaluated in different contexts either in asymptotically simple, asymptotically simple, asymptotically empty or flat spacetime, even if it is future or in the past, so its causal curves are different. There are other types of horizons such as Cauchy and Killing; the first one focuses on the study of surface gravity, spherical bifurcations, acceleration and gravity of bodies with critical mass that converge to be black holes, while the second studies the unextendible causal curves of the past or future, domains and total and partial surfaces of Cauchy evaluating the irreversibility and arrow of time [36]. There is also a cosmological horizon, which, as Nomura says, [37] represents the boundary that limits the region of the universe from which one can still receive signals from deep space, even though it expands, at whose limit there is no spacetime.

The second metric [38] has been formulated to explain the structure of the universe in terms of its expansion, assuming that it is homogeneous and isotropic. In spatial coordinates this type of metric is represented as follows:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sen^2\theta d\emptyset^2\right) \qquad (15)$$

Where k represents a constant in time that describes the curvature and $a(t)$, it is the scale factor of the universe. In order to find the solution to this equation, it is required that normalization rules be used for $a(t_o) = 1$ in which k varies between -1, 0 and 1, or that fits other parameters such as the Hubble constant (70^{+12}_{-8} km/s/Mpc) [39] and the curvature of space.

It should be noted that these proposals of non-Euclidean geometry, give another meaning to the philosophy of contemporary science, where scientific paradigms emerge everywhere, because it is to explain natural phenomena whose behavior and physical manifestations are not conventional, for example , the relation of gravitational lenses with the curvature of spacetime [40], or the phenomenon of quantum teleportation [41], which at present has been given the experimental application that was needed to demonstrate its potential, - apart from the quantum computing, - in creating an encrypted quantum communication network [42], parallel to this great development, it is intended to create and materialize matter through spacetime, which has already been partially done with atoms such as fluorine.

It is important to point out that spacetime has a particular mathematical modeling according to the type of symmetry, that is, it can be stationary or asymptotically flat, static, axially symmetric and spherically symmetric. [33] Therefore, the metrics differ from each other.

Another example of non-Euclidean geometry is the proposal of the superluminal speed of the mathematician Alcubierre, known as *Alcubierre's metric* [43]. This metric raises the possibility of traveling through space at speeds greater than light (known as speed or push Warp), through the manipulation of spacetime, creating a kind of bubble that deforms itself and shrinks in the direction of the apparent movement. In summary, this metric is represented as follows:

$$ds^2 = dx^2 + dy^2 + dz^2 - 2\frac{dx_s(t)}{dt}f\left(r_s(t)\right)dxdt + \left(\left(\frac{dx_s(t)}{dt}\right)^2 f\left(r_s(t)\right)^2 - 1\right)dt^2 \qquad (16)$$

Where

$$r_s(t) = \sqrt{\left(x - x_s(t)\right)^2 + y^2 + z^2} \qquad (17)$$

$$f(r_s) = \frac{tanh(\sigma(r_s+R)) - tanh(\sigma(r_s-R))}{2tanh(\sigma R)} \qquad (18)$$

With the condition that R> 0 and σ> 0 arbitrary. With this type of metric, Alcubierre affirms that a negative energy density is required, which implies the use of some kind of exotic matter that allows to deform spacetime. This metric relates the distances to the square in the space *x*, *y*, *z*, and the temporal distance *t* to the square, which is negative; This sign indicates that it is necessary to measure a distance at a temporal level, -it is a temporal metric- and represents a mathematical object related to the special relativity that demands it. When observing equation (17), what follows is that the body is moving in only one direction, which is along the x axis, where y and z remain unchanged.

On the equation that is constructed with hyperbolic tangent functions, this indicates that a distortion is created at the edges of the volume of flat space, as we have tried to illustrate in figure 10, which is represented by the variation of colors inside-outside.This implies that a new space (such as an expanding universe) is rapidly being created on the back of the moving volume, and the existing space is being annihilated (like a collapsed universe to a Big Crunch) on the front side of the volume moving. [44] A spaceship within the volume described would be propelled forward by the expansion of space that occurs in the back and the contraction of space in front of it.

ISSN 2229-5518

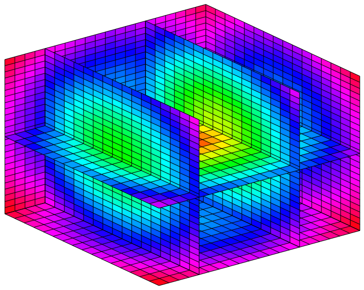


Fig. 10. The simulation shown, is an interpretation of Alcubierre's metric for general relativity, where a body in a region of plane space is surrounded by a distortion, which in this case is blue, that propels it forward to any arbitrary speed, even higher than light. The violet color indicates the surrounding space not deformed.

Although it is a theory, thinking that you can travel in the universe only by deforming space, expanding it, changes the whole context of what is a space trip as it is now, because under the model of Alcubierre, the spacecraft is not the one that moves is the space around it, where the relative time does not change with respect to other systems of reference, and there would be no limit to travel at a speed greater than light, without there being dilations in time.

5 FUTURE INVESTIGATIONS

Due to the dynamics presented by non-Euclidean geometry and its direct relationship with spacetime and time travel back and forth, the research converges to establish which quantum physic models sustained under the Riemannian metrics and Bell inequalities [45], allow explain the phenomenon of remote action or EPR paradox (Einstein, Podolsky, Rosen). The problem of this paradox lies in the phenomenon of non-locality and to define a temporal metric that explains how a system of photons or electrons communicate with each other in situ, when they start from the same source and distance themselves in opposite directions. The paradox lies precisely in the fact that this contradicts the theory of relativity, since it is asserting that information moves at a speed greater than light, hence the name of *action at a distance*.

The research aims to show that it is feasible to have a space-time bubble that forms when the sub-particles move under the phenomenon of distance action on Riemann geodesics that may be incomplete, in which quantum constrictions or singularities are discussed spatio-temporal, suprarelativistic times and quantum multiverses [37], all of them considered under a topology or nonlocal space.

This research covers other fundamental topics for the physical and quantum computing sciences, such as teleportation and physics of the Cúbits, both directly related to the phenomena of local and cosmic quantum entanglement [46]; which has given impetus to quantum computing [47] (local quantum entanglement), which unlike binary systems, works with four states, 11, 00, 10 and 01, so the level of processing and storage of information is extremely fast, in addition to all this is done at atomic scales, so it has no comparison with the current computer systems. For the cosmic scale, it allows encrypted communication between satellites, and what is pursued is to extend it to hundreds of thousands of kilometers, even light years from Earth.

6 CONCLUSION

Euclidean geometry has influenced the world for more than two thousand years, whose contribution to mathematics and science and engineering has been significant, but as it happened with the Aristotelian or Newtonian approaches, which at the time were taken as absolute and dogmatic, that later were rebutted forcing reconstruction and/or reinventing its theoretical and conceptual bases, Euclid's geometry also had something similar, starting with its fifth postulate, which led to other proposals called as non-Euclidean geometries, whose main exponents are Bolyai- Lobachevsky and Riemann, each one with its own characteristics and differentiated one from the other when applied to spacetime. In this sense, it can not be said that Euclid's geometry is excluded from the current context, it only has certain restrictions, like the other geometries, since they work with different metrics, the first on a terrestrial scale, the others on cosmic scales or subatomic. When considering that there are homogeneous non-Euclidean geometries such as Euclidean, elliptical and hyperbolic, the existence of an infinity of possible geometries is considered, which can be described by what is known as general Riemannian varieties [48].

It is interesting to know that the three types of geometry enunciated are valid for the physical world, in which they are used in different contexts. Euclidean geometry, as noted, applies to terrestrial scales, because at higher distances it has errors in terms of the metric that make it unsustainable, especially when you want to have reliable and precise data, particularly the geodesics, which are key to the navigation area and satellite. As for Riemann's geometry, Bolyai and Lobachevsky tried in its time to give a tacit application, which of course was not possible, only until calculations and measurements could be made on macro surfaces by means of so-called geodetic curves; which are described as a line whose length is minimum that joins two points in a spherical geometry surface, and which is also contained in it. The local geodesics on a surface meet the Euclidean axioms, except the fifth.

The non-Euclidean geometries are framed in what Hilbert called metamathematics, who institutionalized it, creating the requisite foundation and formalism, which subsequently extended to various fields of mathematics, logic and physics. With metamathematics, the philosophy of science -very close to geometry- had to change its Kantian position, giving way to new approaches and paradigms that with Euclidean geometry was unsustainable. That is to say, with these changes of the mathematical paradigm, it led to philosophy also doing so in terms of the Kantian conception; which was sustained as stated by Moise and Downs [1] in an absolute assessment of Euclidean geometry. This happened in part, because in the nineteenth century there was an explosion of knowledge in the area of mathematics and mathematical logic, whose contributions to science and engineering have impacted to this day. One aspect to consider about why mathematics diversified so much in this period, obeys the rigorous demonstration models used, which led to rethink in some cases the ancient mathematical basis, including the postulates of Euclid. The implications of the rigor in the old mathematical models, caused that many of his axiomatic expositions, were put under the magnifying glass, like particular case, the fifth postulate of Euclides.

Hilbert worked in diverse contexts of mathematics, in which the theory of numbers [49] and the calculation of variations [50] stand out. The results of this work was the axiomatic reconstruction of Euclidean geometry, which in its principles was considered factual, lasting for millennia. This reconstruction carried out by Hilbert, turns mathematics into a formal axiomatic system. That is to say, Hilbert generalizes the procedure initiated in algebra, to see all mathematics, including analysis, as a set of meaningless formulas, where the propositions of intuitive content of ideal propositions are not distinguished. [51] From Hilbert's point of view, metamathematics is understood syntactically, in which the statements are taken in the demonstrative context of falsehood or truth. The result of this approach, given by Hilbert, is a revolution in axiomatic demonstration models, very different from those proposed by Euclides, which are based more on the factual than on the rigorous. As Giovannini cites [52], in his study of Hilbert's contribution to mathematics and geometry, which clarifies that geometry must be constructed independently of analysis and arithmetic, it coexists with the use of arithmetic and analytic interpretations for show that the various axioms employed are independent of each other.

In general terms, the evolution of non-Euclidean geometries has not stopped, in fact today is more dynamic than ever, especially in research at the level of astrophysics and cosmology, the quantum mechanics of gravitation and the philosophy of mathematics, among others, where recent discoveries show that the nature of spacetime presents a quite particular behavior for n dimensions, example of this, is the string theory, it is even studied about its implications in the possibility of time travel, which is an aspect that has ceased to be science fiction, to become a very important research line of prestigious universities and governments around the world.

To finish, the non-Euclidean geometries are strange, antagonistic and little known to the general public, although we are surrounded by them, for example, the gravitation that comes from the fields of matter and energy like the Sun. This type of geometry is usually worked in the academic, technical and scientific field, in certain areas of knowledge, whose importance is extremely significant for the current and future technological development of society. As for the Euclidean geometry, this has not lost its status, not in terms of the human scale, we live and live with it permanently without in many cases we become aware of it.

REFERENCES

[1] E. Moise y F. Downs. *Geometría moderna.* Colombia: Editorial Addison-Wesley Iberoamericana, 1986.

[2] M. Senior. El surgimiento de las teorías no euclidianas y su influencia en la filosofía de la ciencia del siglo XX. Revista Colombiana de Filosofía de la Ciencia, año/vol. 2, No. 4-5, pp. 45-63, 2001.

[3] J. Gray J. *Ideas de espacio*. Madrid, España. Biblioteca Mondadori, 1992.

[4] P. Lucas. "Las otras geometrías. Conferencia impartida en el curso la Historia de las Matemáticas y su aplicación a la docencia en Enseñanza Secundaria," 1999. [Online]. Available in: http://www.um.es/docencia/plucas/miscelanea/no_euclideas.pdf

[5] F. Vittone. "Introducción a las geometrías no euclideanas." Rosario Differential Geometry Meeting 2012. Rosario University, Rosario, Argentina.

[6] A. V. Trigo, "El quinto postulado de Euclides... y la geometría del universo." Scientific-technical and academic authors. Rev. Acta, pp. 37-46.

[7] S. Gutiérrez y J. Bolyai, "la revolución de la geometría no euclídea," Rev. Suma, febrary, pp. 107-112, 2010.

[8] A. J. Montesinos, "Las geometrías no euclídeas: Gauss, Lobachevsky y Bolyai. Académico numerario," s. f. [Online]. Available in: http://albeniz-matematicas-acaro.wikispaces.com/file/view/las+geometrias+no+euclideas+gauss%2C+lobachevsky+y+bolyai.pdf

[9] A. Ruiz, *"Geometrías no euclidianas. Breve historia de una gran revolución intelectual,"* First edition, Editorial of the University of Costa Rica, 1999.

[10] T. Sochi, *"Introduction to differential geometry of space curves and surfaces,"* First Edition. London. Editorial CreateSpace, 2017.

[11] T. Shifrin, *"Differential geometry: a first course in curves and surfaces,"* United States. Ed. University of Georgia, 2016.

[12] E. J. Márquez, "El plano hiperbólico: historia y fundamentos," Degree thesis, University of Seville. Seville, Spain, 2016.

[13] N. P. Kisbye, El plano de Poincaré. s. f. [Online]. Available in: http://www.famaf.unc.edu.ar/series/pdf/pdfCMat/CMat35-1.pdf

[14] G. F. Etayo, "Matemáticas y realidad. Geometrías no euclídeas y universo," Rev. Acad.Cienc.Exact.Fís.Nat. (Esp), Vol. 104, Nº. 1, pp. 97-105. 2010.

[15] M. Berger, *"A Panoramic View of Riemannian. Geometry,"* Berlin Heidelberg, NewYork. Barcelona HongKong. London. Editorial Springer, 2002.

[16] M. Perdigão do Carmo M, "Geometría Riemanniana," s. f. [Online]. Available in: http://www.nekonekno.x10.mx/doc/doCarmo.pdf

[17] P. L. Hernández, "Introducción a la geometría diferencial," 2002. [Online]. Available in: https://www.unirioja.es/cu/luhernan/lnes.html

[18] J. Lafuente, "Cálculo en variedades," Oct. 2010. [Online]. Available in:

http://www.mat.ucm.es/~jlafuent/own/Manuales/Variedades/cv.pdf

[19] M. Crampon, "Introducción a la geometría diferencial y riemanniana," Escuela Cimpa, Santiago de Chile. 2014. [Online]. Available in: http://members.unine.ch/felix.schlenk/Santiago/cours.pdf

[20] A. Valdés, "Notas de geometría diferencial con aplicaciones," 3 Sep. 2013. [Online]. Available in: https://openlibra.com/es/book/download/notas-de-geometria-diferencial-con-aplicaciones

[21] B. Janssen, "Teoría de la relatividad general," Universidad de Granada. 4to Curso de Física. Granada, España. 22 Sep. 2013. [Online]. Available in: http://www.ugr.es/~bjanssen/text/BertJanssen-RelatividadGeneral.pdf

[22] M. J. Pérez, "Geometría Riemanniana," Universidad de Granada. Dic. 2004. [Online]. Available in: http://www.ugr.es/~jperez/papers/GeomRiem.pdf

[23] R. Albaladejo, "Teoremas de singularidad en relatividad general," Tesis de grado, Universidad de Murcia. Murcia, España, 2016.

[24] L. E. Ibáñez, "Unificación y dualidad en teoría de cuerdas," Rev. Temas 63. Investigación y ciencia. 1er trimestre, pp. 28-34. 2011.

[25] R. Hodson, "The dark universo. Supplement," Rev. Nature, 537, pp. 193. 2016.

[26] L. Tolos, "Exoplanetas con materia oscura," Rev. Investigación y Ciencia, Junio, No, 477 pp. 12-13. 2016.

[27] M. Gorbahn y G. Raffelt, "Viaje a la escala electrodébil. Por qué el Gran Colisionador de Hadrones del CERN deberá encontrar nueva física," Rev. Investigación y ciencia, No. 415, pp. 68-77. 2011.

[28] O Catà, A. Ibarra y S Ingenhütt, "Dark matter decays from non-minimal coupling to gravity," Phys. Rev. Lett. 117, 021302. 2016. https://doi.org/10.1103/PhysRevLett.117.021302.

[29] B. J. Fernández, "Geometría no conmutativa y espaciotiempo cuántico," Rev. Temas 63. Investigación y ciencia. 1er trimestre, pp. 44-53. 2011.

[30] R. G. Hernández, "Geometría Diferencial en el espacio de Minkowski de dimensión tres," Curso de licenciatura Facultad de Ciencias de la UNAM. Ag. 2017. [Online]. Available in: http://www.matem.unam.mx/gruiz/notasminkowski.pdf

[31] E. Witten, "What every physicist should know about string theory," Physics Today 68, 11, pp. 38-43. 2015. https://doi.org/10.1063/PT.3.2980

[32] M. Graham et al., "A possible close supermassive black-hole binary in a quasar with optical periodicity," Rev. Nature 518, pp. 74–76, 2015. https://doi:10.1038/nature14143

[33] E. R. Larrañaga, "Agujeros Negros Clásicos," 28 Ag. 2008. [Online]. Available in: http://www.docentes.unal.edu.co/ealarranaga/docs/AgujerosClasicos.pdf

[34] J. Polanco y F. Arretche, "Sobre las soluciones axiales de vacío y la solución de Schwarzschild," Lat. Am. J. Phys. Educ. Vol. 7, No. 3. 2013

[35] J. Peña, "Agujeros negros: fisuras del espacio tiempo," s. f. [Online]. Available in: http://www.ft.uam.es/personal/rubio/bholes.pdf

[36] A. Matías "Teorías de gravedad, propiedades termodinámicas e irreversibilidad," PhD thesis, Faculty of Exact and Natural Sciences, Department of Physics, University of Buenos Aires, Buenos Aires, Argentina, 2010.

[37] Y. Nomura, "Multiverso Cuántico," Rev. Investigación y ciencia, No 491, pp. 16- 23, 2017.

[38] G. Sáez, "OnFriedmann-Lemaítre-Robertson-Walker cosmologies in non-standard gravity," Tesis de doctorado, Consejo Superior de Investigaciones Científicas, Universitat de Barcelona. Barcelona, España, 2011.

[39] The LIGO Scientific Collaboration and The Virgo Collaboration, The 1M2H Collaboration, The Dark Energy Camera GW-EM Collaboration and the DES Collaboration, The DLT40 Collaboration, The Las Cumbres Observatory Collaboration, The VINROUGE Collaboration & The MASTER Collaboration. "A gravitational-wave standard siren measurement of the Hubble constant," Rev. Nature, pp. 1-6, 2017. https://doi:10.1038/nature24471

[40] T. Kitamura, K. Nakajima and H. Asada, "Demagnifying gravitational lenses toward hunting a clue of exotic matter and energy," Phys. Rev. D 87, 027501. 2013. https://doi.org/10.1103/PhysRevD.87.027501

[41] A. Ulanov et al., "Quantum Teleportation between Discrete and Continuous Encodings of an Optical Qubit," Phys. Rev. Lett. 118, 160501. 2017. https://doi.org/10.1103/PhysRevLett.118.160501

[42] ABC ciencia, "China, a un paso del teletransporte cuántico en el espacio," 20 jun. 2017. [Online]. Available in: http://www.abc.es/ciencia/abci-china-paso-teletransporte-cuantico-espacio-201706152218_noticia.html#disqus_thread

[43] G. Varieschi y Z. Burstein, "Conformal Gravity and the Alcubierre Warp Drive Metric," Hindawi Publishing Corporation, ISRN Astronomy and Astrophysics, Vol. 2013, Article ID 482734, pp. 1-13. http://dx.doi.org/10.1155/2013/482734

[44] G. J. Cramer, "The alcubierre Warp Drive," Analog Science Fiction & Fact Magazine. Nov. 1996. [Online]. Available in: https://www.npl.washington.edu/av/altvw81.html

[45] G. Abal, "Paradoja EPR y desigualdades de Bell: pruebas experimentales, estado actual del conocimiento," Instituto de Física-Facultad de Ingeniería, Universidad de la República, Montevideo, Uruguay, 2007.

[46] J. Handsteiner et al., "Cosmic Bell Test: Measurement Settings from Milky Way Stars," Phys. Rev. Lett. 118, 060401 – Published 7 February 2017. https://doi.org/10.1103/PhysRevLett.118.060401

[47] T. D. Stanescu, *"Introduction to Topological Quantum Matter & Quantum Computation,"* Oakville, Canada, Appel Academic Press In. 2016.

[48] A. Montesdeoca, Apuntes de introducción a las variedades de Riemman. 1998. [Online]. Available in: https://amontes.webs.ull.es/apuntes/geomriem.pdf

[49] F. Bombal, "David Hilbert: La búsqueda de la certidumbre," XV Programa de Promoción de la Cultura Científica y Tecnológica. Rev. Real Acad. Ci. Exactas Físicas y Naturales, Madrid, 123-145. 2013.

[50] M. Krasnov, G. Makarenko y A. Kiseliov, *"Cálculo variacional,"* (Ejemplos y problemas). Madrid, España. Editorial MIR, 1992.

[51] J. Gambra, "La filosofía de David Hilbert," La filosofía de los científicos, pp. 147-179. 1995.

[52] E. Giovannini, "Aritmetizando la geometría desde dentro: el cálculo de segmentos de David Hilbert," scientiæ zudia, São Paulo, v. 13, n. 1, pp. 11-48. 2015.